Functional Dependencies:

- A **functional dependency (FD)** is a relationship between two attributes, X and Y, if for every valid instance of X, that value of X uniquely determines the value of Y. This relationship is denoted as **X → Y**.
  I.e. If column X of a table uniquely identifies column Y of the same table then it can be represented as **X → Y**.
  A functional dependency **X → Y** in a relation holds if two or more tuples having the same value for X also have the same value for Y.
- The left side of the above FD notation is called the **determinant**, and the right side is the **dependent**.
- E.g.
  a. **SIN → Name, Birth date, Address** means that SIN determines Name, Address and Birthdate. Given a SIN, we can determine any of the other attributes within the table.
  b. **ISBN → Title** means that ISBN determines Title.
- **Types of functional dependencies:**
  1. **Multivalued dependency:**
  - **Multivalued dependency** occurs when there are more than one independent multivalued attributes in a table.
  - E.g.

Car_model	Maf_year	Color
H001	2017	Metallic
H001	2017	Green
H005	2018	Metallic
H005	2018	Blue
H010	2015	Metallic
H033	2012	Gray

  In this example, maf_year and color are independent of each other but dependent on car_model. In this example, these two columns are said to be multivalued dependent on car_model.

  This dependence can be represented like this:
  car_model → maf_year
  car_model → colour
  2. **Trivial Functional dependency:**
  - The dependency of an attribute on a set of attributes is known as **trivial functional dependency** if the set of attributes includes that attribute.
  - **Note: A → A** is always a trivial functional dependency.
  - E.g.
    Consider a table with two columns Student_id and Student_Name.

    {Student_Id, Student_Name} → Student_Id is a trivial functional dependency as Student_Id is a subset of {Student_Id, Student_Name}.

    Furthermore, **Student_Id → Student_Id** & **Student_Name → Student_Name** are trivial dependencies too.

3. **Non-trivial Functional Dependency:**

- A **non-trivial functional dependency** occurs when **A → B** where B is not a subset of A.
  I.e. If a functional dependency **X → Y** holds true where Y is not a subset of X then this dependency is called a non-trivial functional dependency.
- E.g.
  Consider an employee table with three attributes: emp_id, emp_name, and emp_address.
  The following functional dependencies are non-trivial:
  emp_id → emp_name (emp_name is not a subset of emp_id)
  emp_id → emp_address (emp_address is not a subset of emp_id)

  However, **{emp_id, emp_name} → emp_name** is trivial because emp_name is a subset of {emp_id, emp_name}.

4. **Transitive Dependency:**

- A functional dependency is said to be **transitive** if it is indirectly formed by two functional dependencies.
- **X → Z** is a transitive dependency if the following three functional dependencies hold true:
  - **X → Y**
  - Y does not → X
  - **Y → Z**
- E.g.

Book	Author	Author_age
Game of Thrones	George R. R. Martin	66
Harry Potter	J. K. Rowling	49
Dying of the Light	George R. R. Martin	66

  {Book} → {Author}
  {Author} does not → {Book}
  {Author} → {Author_age}
  Therefore as per the rule of transitive dependency:
  {Book} → {Author_age} should hold. That makes sense because if we know the book name we can know the author's age.
- **Note:** A transitive dependency can only occur in a relation of three or more attributes.

- **Axioms of functional dependency:**
  1. If we have X → Y and all the values in X are unique, then we know for sure that there is a valid functional dependency between X and Y.
  2. Similarly, if we have X → Y and all the values in Y are the same, then we know for sure that there is a valid functional dependency between X and Y.
  3. **Reflexive Axiom:** If X is a set of attributes and Y ⊆ X, then X → Y.
  4. **Augmentation Axiom:** If X → Y and Z is a set of attributes, then XZ → YZ.
  5. **Transitivity Axiom:** If X → Y and Y → Z, then X → Z.
  6. **Union Axiom:** If X → Y and X → Z, then X → YZ
  7. **Decomposition Axiom:** If X → YZ, then X → Y and X → Z.
  8. **Pseudo Transitivity Axiom:** If X → Y and WY → Z, then WX → Z.

9. **Composition Axiom:** If X → Y and Z → W, then XZ → YW.

Note: The reflexive, augmentation and transitivity axioms are called the **Armstrong Axioms**.

- **Closure/Attribute Closure:**
- Defined as "Given a set of attributes, what are the other attributes that can be fetched from it."
- The closure of an attribute, A, is denoted as A^+.
- **Equivalence of functional dependencies:**
- Let FD1 and FD2 are two FD sets for a relation R.
    1. If all FDs of FD1 can be derived from FDs present in FD2, we can say that FD1 ⊆ FD2.
    2. If all FDs of FD2 can be derived from FDs present in FD1, we can say that FD2 ⊆ FD1.
    3. If 1 and 2 both are true, FD1 = FD2.
- **Irreducible set of functional dependencies/Canonical Form:**
- Whenever a user updates the database, the system must check whether any of the functional dependencies are getting violated in this process. If there is a violation of dependencies in the new database state, the system must roll back. Working with a huge set of functional dependencies can cause unnecessary added computational time. This is where the canonical cover comes into play.
- A canonical cover of a set of functional dependencies F is a simplified set of functional dependencies that has the same closure as the original set F.
- **Examples:**
    1. Given the table and the functional dependencies below, show and explain which functional dependencies are valid and which are invalid.

A	B	C	D	E
a	2	3	4	5
a	2	3	6	5
2	a	3	4	5

A → BC
DE → C
C → DE
BC → A

Soln:

1. A → BC
This one is valid because if you look at the table, under column A, there are 2 a's, and they both correspond to 2 in column B and 3 in column C.

2. DE → C
This one is valid because there are 2 instances of {D: 4, E:5} and they both correspond to 3 in C.

3. $C \rightarrow DE$
   This one is invalid because there are 3 instances of {C:3} but they correspond to different values in DE.
   In the second row, C = 3 corresponds to D = 6 and E = 5 while in rows 1 and 3, C = 3 corresponds to D = 4 and E = 5.

4. $BC \rightarrow A$
   This one is valid because there are 2 instances of B = 2 and C = 3 and both times, they correspond to A = a.

2. Given a relational R with attributes A, B and C, R(A,B,C), and the following functional dependencies, find the closure of A.

   $A \rightarrow B$
   $B \rightarrow C$

   Soln:
   A^+ = {A, B, C} because A can determine A, and B. Furthermore, B can determine C.

3. Given a relational R with attributes A, B, C, D, E, and F, R(A,B,C,D,E,F), and the following functional dependencies, find the closure of D and DE.

   $A \rightarrow B$
   $C \rightarrow DE$
   $AC \rightarrow F$
   $D \rightarrow AF$
   $E \rightarrow CF$

   Soln:
   D^+ = {A, B, D, F} because D can determine A, D and F. Furthermore, A can determine B.

   $(DE)^+$ = {A, B, C, D, E, F} because D can determine A, D and F. Furthermore, A can determine B. E can determine E, C and F.

4. Given R(A, B, C, D, E, F, G) and the following functional dependencies, find the closure of AC.

   $A \rightarrow B$
   $BC \rightarrow DE$
   $AEG \rightarrow G$

   Soln:
   $(AC)^+$ = {A, C, B, D, E} because AC can determine A and C. Then, A can determine B. Then, BC can determine D and E.

5. Given R(A, B, C, D, E) and the following functional dependencies, find the closure of B.

   $A \rightarrow BC$
   $B \rightarrow D$
   $CD \rightarrow E$
   $E \rightarrow A$
   Soln:
   $B^+ = \{B, D\}$ because B can determine B and D.

6. Given R(A, B, C, D, E, F) and the following functional dependencies, find the closure of AB.

   $AB \rightarrow C$
   $BC \rightarrow DE$
   $D \rightarrow E$
   $CA \rightarrow B$

   Soln:
   $(AB)^+ = \{A, B, C, D, E\}$

7. Given R(A, B, C, D, E, F, G, H) and the following functional dependencies, find the closure of BCD.

   $A \rightarrow BC$
   $CD \rightarrow E$
   $E \rightarrow C$
   $D \rightarrow AEH$
   $ABH \rightarrow BD$
   $DH \rightarrow BC$
   $BCD \rightarrow H$

   Soln:
   $(BCD)^+ = \{B, C, D, H, E, A\}$

8. Given R(A, C, D, E, H) and the following 2 sets of functional dependencies
   Set 1:
   $A \rightarrow C$
   $AC \rightarrow D$
   $E \rightarrow ADH$

   Set 2:
   $A \rightarrow CD$
   $E \rightarrow AH$

   We want to know if the 2 sets of functional dependencies are equivalent.

   Soln:
   Step 1: Check if all of the FDs of Set 1 are in Set 2.
   To do so, I will compute the closures of A, AC and E using the functional dependencies of Set 2.

   $A^+ = \{A, C, D\}$ (Knowing A, I can get A, C and D.)
   $(AC)^+ = \{A, C, D\}$ (Knowing A, I can get A, C and D. Knowing C, I can get C.)
   $E^+ = \{E, A, H, C, D\}$ (Knowing E, I can get E, A and H. Knowing A, I can get C and D.)

   Since the FDs of Set 1 are in the closure of each LHS item computed using the FDs of set 2, we know that Set 1 ⊆ Set 2.
   I.e.
   A^+ in Set 1 = {A, C} but A^+ computed using the FDs of Set 2 = {A, C, D}.
   $(AC)^+$ in Set 1 = {A, C, D} but $(AC)^+$ computed using the FDs of Set 2 = {A, C, D}.
   E^+ in Set 1 = {E, A, D, H} but E^+ computed using the FDs of Set 2 = {E, A, H, C, D}.
   Hence, Set 1 ⊆ Set 2.

   Step 2: Check if all of the FDs of Set 2 are in Set 1.
   To do so, I will compute the closures of A and E using the functional dependencies of Set 1.

   $A^+ = \{A, C, D\}$ (Knowing A, I can get A and C. Knowing AC, I can get D.)
   $E^+ = \{E, A, D, H, C\}$ (Knowing E, I can get E, A, D and H. Knowing A, I can get C.)

   A^+ in Set 2 = {A, C, D} but A^+ computed using the FDs of Set 1= {A, C, D}.
   E^+ in Set 2 = {E, A, H} but A^+ computed using the FDs of Set 1 = {E, A, D, H, C}.
   Hence, Set 2 ⊆ Set 1.

   Since Set 1 ⊆ Set 2 and Set 2 ⊆ Set 1, Set 1 = Set 2.

9. Given R(P, Q, R, S) and the following 2 sets of functional dependencies
Set 1:
$P \rightarrow Q$
$Q \rightarrow R$
$R \rightarrow S$

Set 2:
$P \rightarrow QR$
$R \rightarrow S$

We want to know if the 2 sets of functional dependencies are equivalent.

Soln:
Step 1:
$P^+ = \{P, Q, R, S\}$ (Knowing P, I can get P, Q and R. Knowing R, I can get S.)
$Q^+ = \{Q\}$ (Knowing Q, I can get Q.)
$R^+ = \{R, S\}$ (Knowing R, I can get R and S.)
Here, Set 1 $\nsubseteq$ Set 2 because in Set 1, $Q^+ = \{Q, R\}$ while in Set 2, $Q^+ = \{Q\}$.

Step 2:
$P^+ = \{P, Q, R, S\}$ (Knowing P, I can get P and Q. Knowing Q, I can get R. Knowing R, I can get S.)
$R^+ = \{R, S\}$ (Knowing R, I can get R and S.)
Here, Set 2 $\subseteq$ Set 1.

Therefore, Set 2 $\subseteq$ Set 1.

10. Given R(A, B, C) and the following 2 sets of functional dependencies
Set 1:
$A \rightarrow B$
$B \rightarrow C$
$C \rightarrow A$
Set 2:
$A \rightarrow BC$
$B \rightarrow A$
$C \rightarrow A$
We want to know if the 2 sets of functional dependencies are equivalent.

Soln:
Step 1:
$A^+ = \{A, B, C\}$
$B^+ = \{B, A, C\}$
$C^+ = \{C, A, B\}$
Here, Set 1 $\subseteq$ Set 2.
Step 2:
$A^+ = \{A, B, C\}$
$B^+ = \{B, C, A\}$
$C^+ = \{C, A, B\}$
Here, Set 2 $\subseteq$ Set 1.

Therefore, Set 1 = Set 2.

11. Given R(V, W, X, Y, Z) and the following 2 sets of functional dependencies
    Set 1:
    $W \rightarrow X$
    $WX \rightarrow Y$
    $Z \rightarrow WY$
    $Z \rightarrow V$

    Set 2:
    $W \rightarrow XY$
    $Z \rightarrow WX$

    We want to know if the 2 sets of functional dependencies are equivalent.

    Soln:
    Step 1:
    $W^+ = \{W, X, Y\}$
    $(WX)^+ = \{W, X, Y\}$
    $Z^+ = \{Z, W, X, Y\}$
    Here, Set 1 $\nsubseteq$ Set 2. (V is not in Z^+.)

    Step 2:
    $W^+ = \{W, X, Y\}$
    $Z^+ = \{Z, W, Y, V, X\}$
    Here, Set 2 $\subseteq$ Set 1.

    Therefore, Set 2 $\subseteq$ Set 1.

12. Given R(W, X, Y, Z) and the following set of functional dependencies

    $X \rightarrow W$
    $WZ \rightarrow XY$
    $Y \rightarrow WXZ$

    We want to check for redundancy.

    Soln:
    The redundancy can occur at $\propto$, β or $\propto \rightarrow \beta$.

    Step 1: We will remove redundancies at the β level.
    To do this, we will apply the decomposition rule.
    $X \rightarrow W$
    $WZ \rightarrow X$
    $WZ \rightarrow Y$
    $Y \rightarrow W$
    $Y \rightarrow X$
    $Y \rightarrow Z$

Now, we will find the closure of each item on the LHS, first with the FD and second without the FD.

Variable	**With FD**	**Without FD**
X^+	{X, W}	{X} (Without X → W)
$(WZ)^+$	{W, Z, X, Y}	{W, Z, X, Y} (Without WZ → X) **Redundant**
$(WZ)^+$	{W, Z, X, Y}	{W, Z, X} (Without WZ → Y)
Y^+	{Y, W, X, Z}	{Y, X, Z, W} (Without Y → W) **Redundant**
Y^+	{Y, W, X, Z}	{Y, Z, W, X} (Without Y → X) **Redundant**
Y^+	{Y, W, X, Z}	{Y, X, W} (Without Y → Z)

A FD is redundant if it can be recreated some other way.
Hence, the following FDs are redundant:
WZ → X
Y → W
Y → X

The canonical form are the following FDs:
X → W
WZ → Y
Y → Z

Step 2: We will remove redundancies at the ∝ level.
Note: We can't decompose WZ because if we do, we will get different closures.
$(WZ)^+$ = {W, Z, X, Y}
W^+ = {W}
Z^+ = {Z}
Since we can't decompose WZ, nothing changes.

13. Given R(A, B, C, D) and the following set of functional dependencies

$A \rightarrow B$
$C \rightarrow B$
$D \rightarrow ABC$
$AC \rightarrow D$

We want to check for redundancy.

Soln:
Step 1: We will remove redundancies at the β level.
$A \rightarrow B$
$C \rightarrow B$
$D \rightarrow A$
$D \rightarrow B$
$D \rightarrow C$
$AC \rightarrow D$

Variable	With FD	Without FD
A^+	{A, B}	{A} (Without $A \rightarrow B$)
C^+	{C, B}	{C} (Without $C \rightarrow B$)
D^+	{D, A, B, C}	{D, B, C} (Without $D \rightarrow A$)
D^+	{D, A, B, C}	{D, A, C, B} (Without $D \rightarrow B$) **Redundant**
D^+	{D, A, B, C}	{D, A, B} (Without $D \rightarrow C$)
$(AC)^+$	{A, C, D, B}	{A, C, B} (Without $AC \rightarrow D$)

The following FD is redundant:
$D \rightarrow B$

The canonical form are the following FDs:
$A \rightarrow B$
$C \rightarrow B$
$D \rightarrow A$
$D \rightarrow C$
$AC \rightarrow D$

Step 2: We will remove redundancies at the ∝ level.
Note: We can't decompose AC because if we do, we will get different closures.
$(AC)^+ = \{A, C, D, B\}$
$A^+ = \{A, B\}$
$C^+ = \{C, B\}$
Since we can't decompose AC, nothing changes.

14. Given R(V, W, X, Y, Z) and the following set of functional dependencies

$V \rightarrow W$
$VW \rightarrow X$
$Y \rightarrow VXZ$

We want to check for redundancy.

Soln:
Step 1: We will remove redundancies at the β level.
$V \rightarrow W$
$VW \rightarrow X$
$Y \rightarrow V$
$Y \rightarrow X$
$Y \rightarrow Z$

Variable	**With FD**	**Without FD**
V^+	{V, W, X}	{V} (Without $V \rightarrow W$)
$(VW)^+$	{V, W, X}	{V, W} (Without $VW \rightarrow X$)
Y^+	{Y, V, X, Z, W}	{Y, X, Z} (Without $Y \rightarrow V$)
Y^+	{Y, V, X, Z, W}	{Y, V, Z, W, X} (Without $Y \rightarrow X$) **Redundant**
Y^+	{Y, V, X, Z, W}	{Y, V, X, W} (Without $Y \rightarrow Z$)

The following FD is redundant:
$Y \rightarrow X$

The canonical form are the following FDs:
$V \rightarrow W$
$VW \rightarrow X$
$Y \rightarrow V$
$Y \rightarrow Z$

Step 2: We will remove redundancies at the ∝ level.
$(VW)^+$ = {V, W, X}
V^+ = {V, W, X}
W^+ = {W}

Hence, we can rewrite $VW \rightarrow X$ into $V \rightarrow X$.

The canonical form are the following FDs:
$V \rightarrow W$
$V \rightarrow X$
$Y \rightarrow V$
$Y \rightarrow Z$

Tutorial Notes:

E.g. 1. Determine if the 2 sets of FDs are equivalent.

Set 1
A → BC

Set 2
A → B
A → C

Soln:
Step 1: Check if all the FDs of Set 1 are in Set 2.
To do so, compute the closure of A using the FDs of Set 2.
A^+ = {A, B, C}
We can see that A^+ in Set 1 is a subset of A^+ computed using the FDs of Set 2.
Hence, Set 1 ⊆ Set 2.

Step 2: Check if all the FDs of Set 2 are in Set 1.
To do so, compute the closure of A using the FDs of Set 1.
A^+ = {A, B, C}
We can see that A^+ in Set 2 is a subset of A^+ computed using the FDs of Set 1.
Hence, Set 2 ⊆ Set 1.

Since Set 1 ⊆ Set 2 and Set 2 ⊆ Set 1, Set 1 = Set 2, meaning they are equivalent.

E.g. 2. Suppose the FD BC → D holds for relation R. Create an instance of relation R that breaks the FD. R(A, B, C, D)

Soln:

A	B	C`	D
1	1	2	3
2	1	2	4

We see that there are 2 instances of 1|2 for B|C, but their D values are different. Hence, we cannot use B|C to uniquely determine D.

E.g. 3. Determine if the 2 sets of FDs are equivalent.

Set 1
PQ → R

Set 2
P → R
Q → R

Soln:
Step 1: Get the closure of PQ using the FDs from Set 2.
$(PQ)^+$ = {P, Q, R}

Step 2: Get the closure of P and Q using the FDs from Set 1.
$(P)^+ = \{P\}$
$(Q)^+ = \{Q\}$

We can see that Set 1 ⊆ Set 2 but Set 2 ⊈ Set 1.
Hence, they are not equivalent.

E.g. 4. Determine if the 2 sets of FDs are equivalent.

Set 1
$PQ \rightarrow R$

Set 2
$P \rightarrow Q$
$P \rightarrow R$

Soln:
Step 1: Get the closure of PQ using the FDs from Set 2.
$(PQ)^+ = \{P, Q, R\}$

Step 2: Get the closure of P using the FDs from Set 1.
$(P)^+ = \{P\}$

We can see that Set 1 ⊆ Set 2 but Set 2 ⊈ Set 1.
Hence, they are not equivalent.

E.g. 5. Given R(A, B, C, D, E, F) and the FDs
$AC \rightarrow F$
$CEF \rightarrow B$
$C \rightarrow D$
$DC \rightarrow A$

a. Does the FD $C \rightarrow F$ hold?

   Soln:
   Another way of thinking about this is “Does the closure of C include F?”
   The closure of C = {C, D, A, F}.
   Yes, $C \rightarrow F$ holds.

b. Does the FD $ACD \rightarrow B$ hold?

   Soln:
   The closure of ACD = {A, C, D, F}.
   Hence, $ACD \rightarrow B$ does not hold.

Projection:
E.g. 1. Given R(A, B, C, D, E) and the FDs
$A \rightarrow C$
$C \rightarrow E$
$E \rightarrow BD$

Project these FDs onto R1(A, B, C).

Soln:
Let's find the closure of A, B and C.
$A^+ = \{A, C, E, B, D\}$ However, since R1 only has the attributes A, B and C, we get the FD $A \rightarrow BC$. We ignore $A \rightarrow A$ because it's a trivial FD.
$B^+ = \{B\}$ This is a trivial FD, so we ignore it.
$C^+ = \{C, E, B, D\}$ However, since R1 only has the attributes A, B and C, we get the FD $C \rightarrow B$.

The projection of the FDs onto R1 is $\{A \rightarrow BC, C \rightarrow B\}$.

E.g. 2. Given R(A, B, C, D, E) and the FDs
$A \rightarrow C$
$C \rightarrow E$
$E \rightarrow BD$

Project these FDs onto R1(A, D, E).

Soln:
Let's find the closure of A, D and E.
$A^+ = \{A, C, E, B, D\}$ However, since R1 only has the attributes A, D and E, we get the FD $A \rightarrow DE$. We ignore $A \rightarrow A$ because it's a trivial FD.
$D^+ = \{D\}$ This is a trivial FD, so we ignore it.
$E^+ = \{E, B, D\}$ However, since R1 only has the attributes A, D and E, we get the FD $E \rightarrow D$.

The projection of the FDs onto R1 is $\{A \rightarrow DE, E \rightarrow D\}$.

E.g. 3. Given R(A, B, C) and the FDs
$A \rightarrow B$
$B \rightarrow C$

Project these FDs onto R1(A, C).

Soln:
Let's find the closure of A and C.
$A^+ = \{A, B, C\}$ However, since R1 only has the attributes A and C, we get the FD $A \rightarrow C$. We ignore $A \rightarrow A$ because it's a trivial FD.
$C^+ = \{C\}$ This is a trivial FD, so we ignore it.

The projection of the FDs onto R1 is $\{A \rightarrow C\}$.

E.g. 4. Given R(A, B, C, D) and the FDs
$A \rightarrow B$
$B \rightarrow C$
$C \rightarrow D$

Project these FDs onto R1(A, C, D).

Soln:
Let's find the closure of A, C, and D.
$A^+ = \{A, B, C, D\}$ However, since R1 only has the attributes A, C and D, we get the FDs $A \rightarrow C$ and $A \rightarrow D$. We ignore $A \rightarrow A$ because it's a trivial FD.
$C^+ = \{C, D\}$ However, since R1 only has the attributes A, C and D, we get the FD $C \rightarrow D$. We ignore $C \rightarrow C$ because it's a trivial FD.
$D^+ = \{D\}$ This is a trivial FD, so we ignore it.

The projection of the FDs onto R1 is $\{A \rightarrow C, A \rightarrow D$ and $C \rightarrow D\}$.

E.g. 5. Given R(A, B, C, D, E, F) and the FDs
$A \rightarrow BC$
$C \rightarrow DE$
$E \rightarrow A$

Project these FDs onto R1(A, C, E).

Soln:
Let's find the closure of A, C, and E.
$A^+ = \{A, B, C, D, E\}$ However, since R1 only has the attributes A, C and E, we get the FDs $A \rightarrow C$ and $A \rightarrow E$. We ignore $A \rightarrow A$ because it's a trivial FD.
$C^+ = \{C, D, E, A\}$ However, since R1 only has the attributes A, C and E, we get the FDs $C \rightarrow E$ and $C \rightarrow A$. We ignore $C \rightarrow C$ because it's a trivial FD.
$E^+ = \{E, A, B, C, D\}$ However, since R1 only has the attributes A, C and E, we get the FDs $E \rightarrow A$ and $E \rightarrow C$. We ignore $E \rightarrow E$ because it's a trivial FD.

The projection of the FDs onto R1 is $\{A \rightarrow C, A \rightarrow E, C \rightarrow A, C \rightarrow E, E \rightarrow A$ and $E \rightarrow C\}$.